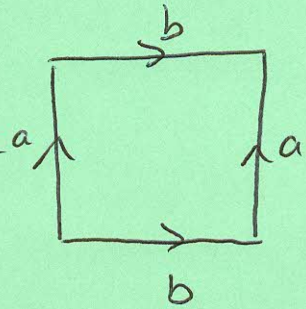
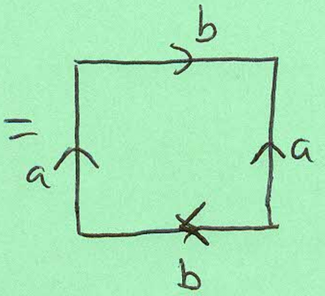
Consider the following:

$\mathbb{T} =$ 

$\Rightarrow \chi(\mathbb{T}) = 1-2+1=0$

$\mathbb{K} =$ 

$\Rightarrow \chi(\mathbb{K}) = 1-2+1=0$

The Euler characteristic cannot distinguish between these two, but we know they're different because $\mathbb{T}$ is orientable while $\mathbb{K}$ isn't. It turns out that the Euler characteristic plus knowing the orientability is enough

Classification Theorem for Closed Surfaces

Let Σ be a closed, connected surface. Then Σ is homeomorphic to one of:

- S^2
- $g\mathbb{T} = \underbrace{\mathbb{T} \# \cdots \# \mathbb{T}}_{g}$
- $m\mathbb{P} = \underbrace{\mathbb{P} \# \cdots \# \mathbb{P}}_{m}$

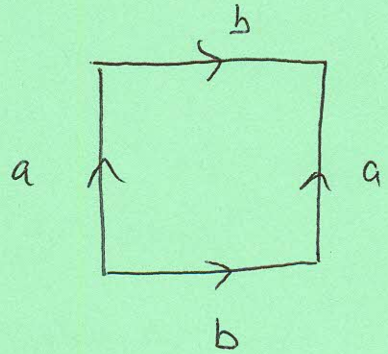
This brings us to the questions:

$$\chi(g\mathbb{T}) = ?$$

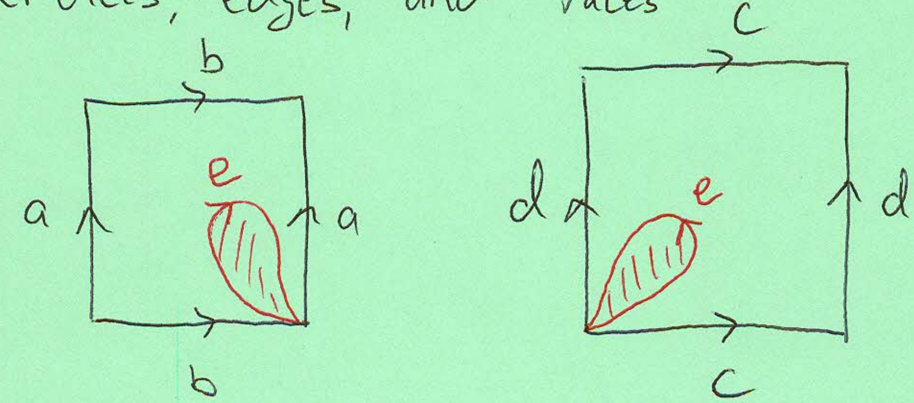
$$\chi(m\mathbb{P}) = ?$$

Let's revisit $\mathbb{T} \# \mathbb{T}$:

$\mathbb{T}$ has $V=1$, $E=2$, $F=1$ if we use



Let's form $\mathbb{T} \# \mathbb{T}$ and keep track of vertices, edges, and faces:

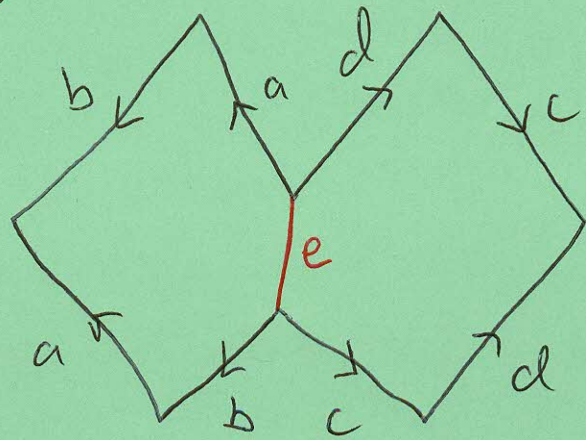


$V_1 = 1$ $\quad V_2 = 1$

$E_1 = 2 + 1$ adding e creates a face $\quad E_2 = 2 + 1$

$F_1 = 1 + 1 - 1$ ← but we cut out that face $\quad F_2 = 1 + 1 - 1$

Now, splitting them at the vertex w/ e, then gluing the two together along e :



$V = 1$

$E = 5 \quad \Rightarrow \chi(\mathbb{T} \# \mathbb{T}) = 1 - 5 + 2 = -2$

$F = 2$

How does $\chi(M \# N)$ relate to $\chi(M)$ and $\chi(N)$?

If M has V_1 vertices, E_1 edges, and F_1 faces and N " V_2 " , E_2 " , " F_2 " , then by cutting out a disk in both surfaces gives +1 edges to each. Then, we glue along the edges created by the cutout disks. This combines the numbers of vertices, edges, and faces, but we lose one edge total because they have been combines (one from each surface) and we also lose one vertex total for the same reason (the vertex is the one that the edge of the cut-out disk starts and ends at). Thus, $M\#N$ has

$V_1 + V_2 - 1$ vertices

$E_1 + 1 + E_2 + 1 - 1$ edges

$F_1 + F_2$ faces

So,

$$\chi(M\#N) = (V_1 + V_2 - 1) - (E_1 + E_2 + 1) + (F_1 + F_2)$$

$$= \chi(M) + \chi(N) - 2$$

<u>Theorem</u>: If $\chi(M) = m$ and $\chi(N) = n$, then

$$\chi(M\#N) = m + n - 2.$$

We can then find that:

$$\chi(g\mathbb{T}) = 2-2g$$

(HW)

and $$\chi(m\mathbb{P}) = 2-m$$

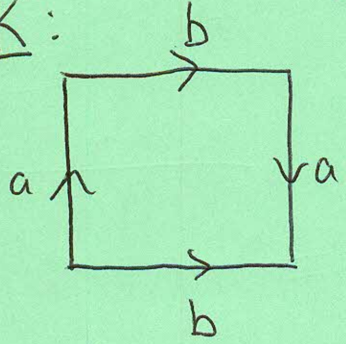
So, how do we identify surfaces?

Only $\chi(m\mathbb{P})$ has the potential of being odd, so if the Euler characteristic is ever odd, it is $m\mathbb{P}$, for some m. But if the Euler characteristic is even...

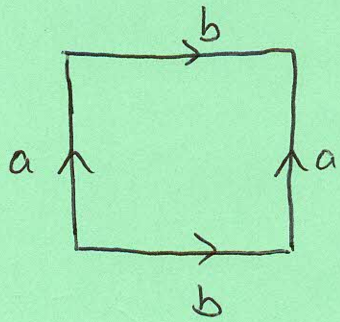
Let's revisit $\mathbb{T}$ & $\mathbb{K}$: both have E.C. $= 0$.

(The following is unique to surfaces).

$\underline{\mathbb{K}}$:



$\underline{\mathbb{T}}$:



Note that both have $\underline{\underline{1}}$ vertex.

Let's read the "word" around the edge

$\mathbb{K}$: $abab^{-1}$ $\mathbb{T}$: $aba^{-1}b^{-1}$

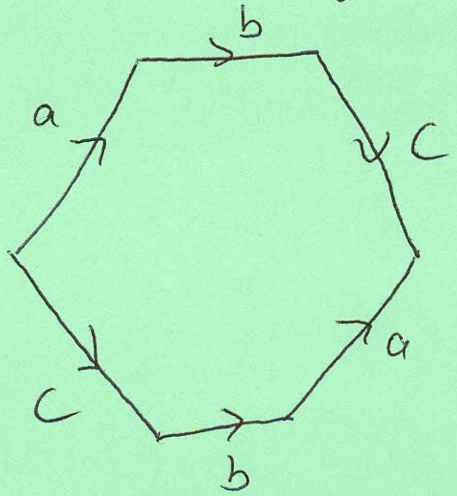
(Inverse when we go against the arrow.)

Notice that in $\mathbb{T}$, every letter comes with an inverse pair: a&a^{-1}, b&b^{-1}; but in $\mathbb{K}$, a does not have an inverse pair. This is how we detect non-orientability, we read the word from the polygon, and check to see if any letter does not have a corresponding inverse letter. This relys on the polygon representation having only one vertex, The next question is:

Q: Can every closed surface be represented as a polygon with edge identifications? With only one vertex?

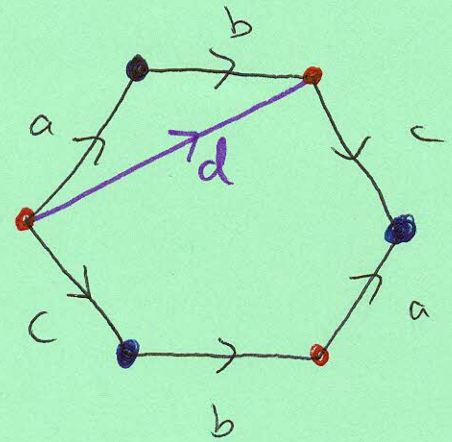
A: Yes and yes. The first question is VERY hard, so we won't prove it here.

Ex: Which surface is given by

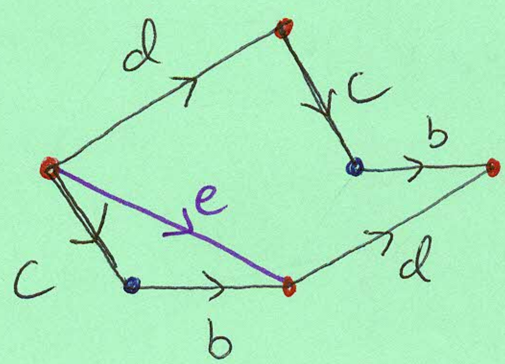


?

Sol: Notice this has 2 vertices: red and blue

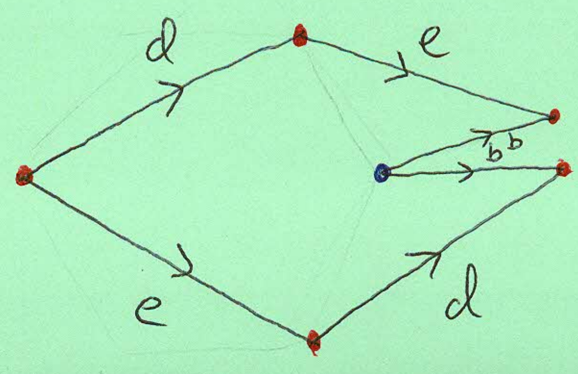


Cut along d
glue along a :

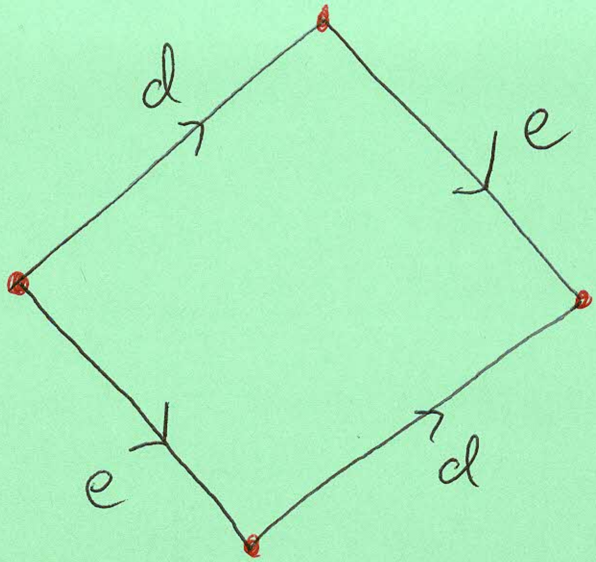


There are still two vertices, but one less blue one.

Cut along e
glue along c :



Again, one less blue vertex.
Now glue the b edges together:



This we recognize as a torus!